Yau Mathematical Competition 2018
Probability and Statistics Overall

Problem 1 (Probability) Let X be a real valued random variable such that

$$\mathbb{E}f(X+1) = \mathbb{E}(Xf(X))$$

for all smooth function f with compact support.

(1) Show that X has a rapidly vanishing positive tail probability, i.e.,

$$\mathbb{P}\{X > N\} \leq \frac{1}{N!}.$$

(2) Use (1) or otherwise to show that X has the standard Poisson distribution $P(1)$.

Problem 2 (Statistics) Let $X_1, \cdots, X_m$ be a random sample from a $N(\mu_1, \sigma_1^2)$ distribution and let $Y_1, \cdots, Y_n$ be another random sample from $N(\mu_2, \sigma_2^2)$. Further, assume that

(a) $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are independent;

(b) $m \geq 2$ and $n \geq 2$;

(c) μ_1 and μ_2 are real, and σ_1 and σ_2 are positive.

Finally, the parameters are all unknown.

(1) How to test $H_0 : \sigma_1^2 \leq \sigma_2^2$ versus $H_1 : \sigma_1^2 > \sigma_2^2$?

(2) If we cannot assume that the distributions for the X_i's and Y_i's are normal, how would you test whether the two samples come from the same distribution?